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Half-Range Expansion Analysis for Langevin Dynamics in the High-Friction Limit with a Singular Absorbing Boundary Condition: Noncharacteristic Case

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We consider the dynamics of a Brownian particle given by the Langevin equation in a strip, under the effects of a deterministic force. The trajectories of particles originate at a source whose spatial location in the phase space coincides with the location of adsorbing boundaries. This leads to singular behavior of trajectories in the high-friction limit. We use the half-range expansion technique and systematic asymptotics to solve a boundary value problem for the Fokker–Planck operator and to calculate the steady-state transition probability density, the mean time to absorption, and the distribution of exit points. We do not make assumptions about other parameters in the problem except that they are $O(1)$ relative to the friction coefficient. We calculate explicitly the correct location of the Milne-type extrapolation for absorbing boundary conditions for the Smoluchowski approximation to the Langevin equation.

KEY WORDS: Langevin equation; Smoluchowski equation; half-range expansion; singular perturbations; Milne extrapolation length.

1. INTRODUCTION

In this paper we derive the leading-order approximation to a problem for a diffusion process given by the Langevin equation in a slab, in the presence of absorbing boundary conditions, in the asymptotic limit of high friction. We treat the problem in which all trajectories of the diffusion process originate at a source whose spatial location in the phase space coincides with the spatial location of an absorbing boundary. In the high-friction limit the behavior of the trajectories of the Langevin process in

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the neighborhood of such boundaries becomes singular. We conduct an asymptotically exact analysis, with no *ad hoc* assumptions about the form of the solution. We include the effects of a potential field. We also consider the problem in which trajectories of the process originate away from absorbing boundaries.

It is well known that the Langevin operator approaches the Smoluchowski operator in the limit of limit friction.[1–3] That is, in the high-damping regime, the dynamics in the phase space of position and velocity can be approximated by dynamics of the position only. This projection of the two-dimensional space onto a one-dimensional space is constructed putting no conditions on the boundary behavior of the underlying process. Thus the Smoluchowski operator is the free-space limit for the Langevin operator in the high-damping regime.

Both the Smoluchowski and the Langevin equations have been extensively used to model many physical and chemical phenomena.[2,4] In many situations the starting point for modeling has been the Langevin equation; in other situations (e.g., particle motion in the liquid phase), the starting point for the modeling has been the simpler Smoluchowski equation. In this paper, we discuss level-crossing problems for dynamics of a Brownian particle in a force field, given by the Langevin equation in the asymptotic limit of high friction. For the class of level-crossing problems, physical boundary conditions are easily translated into the phase space of the Langevin equation; however, it has been an outstanding classical problem to convert them into boundary conditions in the lower-dimensional space of the Smoluchowski equation.[5]

There are many applications in which problems of this form are a natural expression of the underlying physics. The problem considered here is concerned with transport of particles from a source to a receptor. It is a special case of a Boltzmann equation, with a differential scattering operator.[6,7] Many kinds of spectroscopy can only provide information about part of the state of a particle (x, but not $\dot{x}$, for example). In such a situation, the experiment provides information about the solution to level-crossing problems (mean time to travel from $x = x_0$ to $x = x_1$, for example).[4]

Stochastic models of transport in ionic channels require the solution to level-crossing problem.[9] Consider the situation when a source of ions is placed at position $x = 0$ (see Fig. 1). The velocity distribution of the source is given. The source emits ions with positive and negative velocities. Those with negative velocities are immediately resorbed, and those with positive velocities are followed until they are absorbed either on the left (when they return to the level $x = 0$ where source is located), or on the right (when they get to a given level where another absorbing boundary is located).

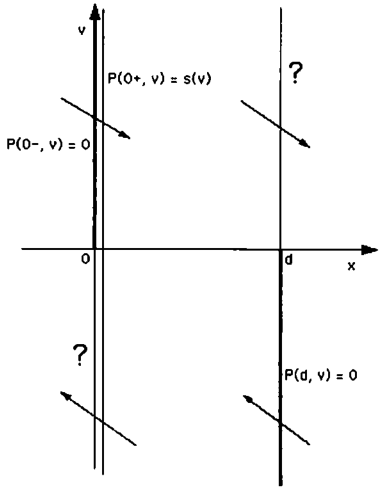

Fig. 1. Absorbing boundary conditions $P=0$ are prescribed on $\{x=0^-, v>0\}$ and $\{x=d, v<0\}$, where the flow is in. The source with the velocity density $s(v)$ is placed at $x=0^+$. The values P takes on $\{x=0^-, v<0\}$ and $\{x=d, v>0\}$, where the flow is out, are to be determined by solving the problem (2.10)–(2.12).

Since ions can reach the boundary on the left only with negative velocities, the absorbing boundary condition can only be imposed on the $\dot{x}>0$ half-line to ensure that no other ions than those originated by the source enter the domain. Similarly, ions can reach the boundary on the right only with positive velocities, so absorbing boundary conditions are imposed on the $\dot{x}<0$ half-line. The flux of exiting ions on the left and on the right is determined by the intensity of the source and the dynamics in the strip between the boundary lines.

In this model, the boundary at $x=0$, $\dot{x}>0$ is singular, since it is both the source of particles and an absorber of particles. Special care will be used below to treat this type of boundary condition. In the process, the correct treatments for simpler nonsingular absorbing boundary conditions will also be developed.

Naive considerations at $x=0$ lead to contradiction. Suppose that one wants to approximate the Langevin operator in the strip by the Smoluchowski operator on an interval. Then the Smoluchowski operator should be equipped with adsorbing boundary conditions on both ends of the interval, together with the source of trajectories located on the left. Thus the boundary on the left would be at the same time an absorbing boundary and the source for trajectories of a diffusion process. Thus no trajectories would ever enter the interval. It is obvious that such an approximation cannot provide the correct asymptotic behavior of the original Langevin problem in the high-friction limit. In this setup with the same location of the source and an absorbing barrier it is evident that the problem is of singular perturbation type.

In this paper we study the boundary value problem for the Fokker–Planck operator associated with the Langevin equation, with absorbing boundary conditions on half-lines, $\dot{x} > 0$ on the left and $\dot{x} < 0$ on the right. We assume that both boundaries are noncharacteristic, that is, the force field does not vanish at the boundaries. Two problems will be solved: one with the singular boundary boundary conditions described above, and a simpler one with regular boundary conditions, where the source and the absorbing boundaries are well separated. We derive the steady-state transition probability density and employ it to calculate the mean first time to absorption and the probability flux at the boundaries. We derive the correct absorbing boundary conditions for the Smoluchowski approximation.

Similar problems, where the external conditions may only be prescribed on part of the boundary, have been addressed previously. The classical problem, first posed in ref. 8, deal with the level crossing of the trajectories of the Ornstein–Uhlenbeck process with no external forcing term. This problem has been the subject of many studies[10–17] because it is the simplest nontrivial formulation of a class of transport phenomena. Moreover, modeling of various physical situations leads to formulations which can be interpreted in the language of the Ornstein–Uhlenbeck process; see, for example, ref. 12. The Wang–Uhlenbeck problem[8] has been solved recently.[18,20,21] Equivalently, it is a problem for a level crossing of trajectories of a Brownian particle in a half-space, under no deterministic external forcing term. The boundary value problem for the transition probability density function is then formulated in terms of the Fokker–Planck operator. The analytical difficulty of solving this problem stems from the fact that half the set of eigenfuntions for the Fokker–Planck operator does not satisfy orthogonality conditions on the part of the boundary where the conditions are imposed, although this half set is complete on this of the boundary.[22] The earlier attempts on the problem relied on arbitrary postulates on the form of the solution at the boundary, for example, the form of the current at the boundary,[16] moment closure assumptions,[15,17] numerical calculations,[10,13] *ad hoc* correction factors.[11] These analyses provided some approximation to the Milne extrapolation length.

The effect of a linear potential on a Brownian particle with absorbing boundary conditions on the half-line $x = 0$, $\dot{x} > 0$ was studied in ref. 19. The problem was considered on the half-space $x > 0$ with an additional zero condition imposed on the solution as $x \to \infty$. The analysis of the stationary problem for a particle originating at $x = 0$ was conducted and the probability of return to the origin was calculated. The Milne extrapolation length was calculated when the particle originates away from the boundary, in the limit of no potential.

The problem solved in this paper is different from and much more general than the problem considered in ref. 19. We do not assume that the potential is linear. We only require that the force does not vanish at the boundary. Our results include as a special case those of ref. 19. We consider the problem on a slab, that is, on a bounded interval for the spatial variable x. The solution of ref. 19 roughly corresponds to our boundary layer solution, but with a significant difference. In our case the boundary layer solution approaches the nonzero outer solution. Moreover, we solve two problems for the Milne extrapolation length. First we calculate the location of the extrapolated boundaries in the singular case when the location of the absorbing boundaries coincides with the location of the source. We also calculate the Milne length in the regular case. In both cases we derive the effect of the potential on the extrapolated boundaries. We calculate the probability of return to the origin for an arbitrary potential and the slab geometry. In particular, we specialize our results to a linear potential and if we extend the thickness on the slab to infinity then we recover the results of ref. 19.

In refs. 20 and 21 a technique of expanding the solution in half the set of eigenfunctions, so that the conditions given on the half of the boundary are satisfied, was developed—the technique of half-range expansion. We generalize and use this technique. We solve the boundary value problems using a systematic expansion in the limit of high friction, with no need for assumptions about the solution. In the case of a linear potential, we derive the complete asymptotic expansion, yielding a solution accurate to within a transcendentally small eror. Our two principal tools are half-range expansions and boundary-layer expansions. The completeness results in refs. 22–24 provide the theoretical basis for the half-range expansions we use at the boundaries.

In Section 2, we formulate the problem. In Section 3, we present the main results of this paper: (i) the expression for the steady-state probability density, uniformly valid in the domain; (ii) properties of the original process, such as the mean first passage time to the boundary, and the probability flux on the boundary; and (iii) the correct boundary conditions for the Smoluchowski approximation, in both the regular and singular cases. In Section 4, preliminary technical results are developed, which are applied in Sections 5 and 6 to derive the solutions to the problems. (The technique developed and applied herein may also be used to solve the full time-dependent problem.)

2. FORMULATION OF THE PROBLEM

We consider a diffusion process $(x, v \equiv \dot{x})$ whose trajectories obey the Langevin equation

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\beta v - U'(x) + (2\beta)^{1/2}\,\delta \dot{w} \end{aligned} \tag{2.1}$$

Here β denotes the damping coefficient, δ^2 is the noise strength, $U(x)$ is a deterministic potential, and $\dot{w}$ is standard Gaussian white noise.

We consider the process (x, v) on the strip $D \equiv [0, d] \times \mathscr{R}$; trajectories originate at $x = 0^+$, with a known velocity distribution $s(v)$. No other trajectories enter D. Thus the transition density $p(x, v, t \mid \xi = 0^+, \eta)$ satisfies the problem given by the Fokker–Planck operator $L_{x,v}$,

$$\frac{\partial p}{\partial t} = L_{x,v}\, p \equiv -v \frac{\partial p}{\partial x} + \frac{\partial}{\partial v}\left[(\beta v + U'(x))\, p \right] + \delta^2 \beta \frac{\partial^2 p}{\partial v^2} \tag{2.2}$$

with the initial condition

$$\lim_{t \to 0^+} p(x, v, t \mid \xi = 0^+, \eta) = \delta(x - 0^+)\, \delta(v - \eta)\, s(\eta) \tag{2.3}$$

and the absobing boundary conditions

$$p(x = 0^-, v, t \mid \xi = 0^+, \eta) = 0 \qquad \text{for} \quad v > 0 \tag{2.4}$$

$$p(x = d, v, t \mid \xi = 0^+, \eta) = 0 \qquad \text{for} \quad v < 0 \tag{2.5}$$

The absorbing boundary conditions indicate that trajectories, once they reach the boundary of D, are terminated so they do not reenter D. Moreover, Eqs. (2.3) and (2.4) show that the density p suffers a discontinuity at $x = 0$ and $v > 0$, since $p \equiv 0$ for $x < 0$. This singular behavior of the solution requires a special technique to be developed to obtain the solution to the problem (2.2). The analysis will be carried out for a general $s(\eta)$; we will specifically give results for the Maxwellian density

$$s(\eta) = \frac{1}{(2\pi)^{1/2}\delta}\, e^{-\eta^2/2\delta^2} \tag{2.6}$$

which describes the source in the state of thermal equilibrium. We solve the problem (2.2)–(2.5) in the asymptotic limit of $\beta \gg 1$. The noise strength δ^2 is not assumed to be small compared to other parameters of the problem, that is, compared to $U(x)$ and $U'(x)$ on $(0, d)$. We assume that $x = 0$ and

$x=d$ are noncharacteristic boundaries for the potential $U(x)$, that is, $U'(0) \neq 0$ and $U'(d) \neq 0$. The special case of $U(x) \equiv 0$ is solved in Section 6. The case of characteristic boundaries will be considered separately.[25]

First we introduce a new time scale

$$\hat{t} = t/\beta \tag{2.7}$$

and rewrite Eq. (2.2) as

$$\frac{1}{\beta}\frac{\partial p}{\partial \hat{t}} = L_{x,v}\, p \tag{2.8}$$

We define the function

$$P(x, v \mid \xi = 0^+) = \int_{-\infty}^{+\infty} \left[\int_0^{+\infty} p(x, v, \hat{t} \mid \xi = 0^+, \eta)\, d\hat{t} \right] d\eta \tag{2.9}$$

The function βP can be interpreted as the mean time spent by a trajectory at the point (x, v) prior to its absorption, given that it started at position ξ.[26] It follows from (2.2)–(2.5) that P satisfies the problem

$$L_{x,v} P = -\frac{1}{\beta}\,\delta(x - 0^+)\, s(v) \tag{2.10}$$

$$P(x = 0^-, v \mid \xi = 0^+) = 0 \qquad \text{if} \quad v > 0 \tag{2.11}$$

$$P(x = d, v \mid \xi = 0^+) = 0 \qquad \text{if} \quad v < 0 \tag{2.12}$$

(See Fig. 1.) Knowing P, we can calculate properties of the process (x, v), for example,

(i) the mean first passage time (the mfpt), $E\tau$, given that the trajectory started at $\xi = 0^+$, is given by

$$E\tau = \beta \iint_D P(x, v \mid \xi = 0^+)\, dx\, dv \tag{2.13}$$

(ii) the density of exit points is given by

$$p(x(\tau) = 0, v(\tau) \in (v, v + dv) \mid x(0) = 0^+) = -\beta v P(x = 0, v \mid \xi = 0^+)\, dv \qquad \text{if} \quad v < 0 \tag{2.14}$$

$$p(x(\tau) = d, v(\tau) \in (v, v + dv) \mid x(0) = 0^+) = \beta v P(x = d, v \mid \xi = 0^+)\, dv \qquad \text{if} \quad v > 0 \tag{2.15}$$

(iii) the total probability flux at $x=0$, $\Psi(0 \mid \xi=0^+)$, and the total propability flux at $x=d$, $\Psi(d \mid \xi=0^+)$, are given by

$$\Psi(0 \mid \xi=0^+) = -\beta \int_{-\infty}^{0} vP(x=0, v \mid \xi=0^+)\, dv \tag{2.16}$$

$$\Psi(d \mid \xi=0^+) = \beta \int_{0}^{\infty} vP(x=d, v \mid \xi=0^+)\, dv \tag{2.17}$$

In order words, $\Psi(d \mid \xi=0^+)$ equals to the probability that trajectories exit D at $x=d$ (with any velocity) given that they originated at $\xi=0^+$. For future reference, we write the relation between $\Psi(0 \mid \xi=0^+)$ and $\Psi(d \mid \xi=0^+)$,

$$\Psi(0 \mid \xi=0^+) = \tfrac{1}{2} + [\tfrac{1}{2} - \Psi(d \mid \xi=0^+)] \tag{2.18}$$

The expression in the brackets denotes the flux of the trajectories at $x=0$, $v<0$, which originated at $x=0$, $v>0$. The factor 1/2 to the right of the equality sign indicates that half of the trajectories which originated at the source do not enter D; they contribute to the probability flux at $x=0$, just before they are terminated. [Here the factor 1/2 follows from the assumption that $\int_{\infty}^{0} s(v)\, dv = \int_{0}^{\infty} s(v)\, dv = 1/2$, that is, half of the trajectories originate with negative velocities. If this is not the case, then the factor 1/2 in the brackets should be replaced by the proportion of the trajectories with negative velocities, and the factor 1/2 to the right of the equality sign should be replaced by the proportion of the trajectories with positive velocities.]

We seek the soluton to (2.10) in the form

$$P(x, v \mid 0^+) = \frac{1}{(2\pi)^{1/2}\delta} e^{-v^2/2\delta^2} e^{-U(x)/\delta^2} Q(x, v \mid \xi=0^+) \tag{2.19}$$

Next we define

$$\hat{s}(v) \equiv (2\pi)^{1/2} \delta e^{v^2/2\delta^2} s(v) \tag{2.20}$$

and we rescale the velocity

$$\hat{v} \equiv v/\delta \tag{2.21}$$

Also we denote $\varepsilon \equiv 1/\beta$. Upon dropping the hat, we obtain the problem for Q:

$$\mathscr{L}_{x,v} Q(x, v) \equiv Q_{vv} + \left(-v + \varepsilon \frac{U'(x)}{\delta} \right) Q_v - \varepsilon \delta v Q_x$$

$$= -\varepsilon^2 e^{U(x)/\delta^2} s(v)\, \delta(x - 0^+)$$

$$Q(x = 0^-, v \mid \xi = 0^+) = 0 \quad \text{if} \quad v > 0 \tag{2.22}$$

$$Q(x = d, v \mid \xi = 0^+) = 0 \quad \text{if} \quad v < 0$$

Below we solve the problem (2.22) in the asymptotic limit of $\varepsilon \ll 1$.

We write the formulas for the mfpt and the total flux at the boundary in terms of the function Q as

$$E\tau = \frac{1}{\varepsilon} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \int_0^d e^{-v^2/2} e^{-U(x)/\delta^2} Q(x, v)\, dx\, dv \tag{2.23}$$

$$\Psi(0) = -\frac{\delta}{\varepsilon} e^{-U(0)/\delta^2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} v e^{-v^2/2} Q(0, v)\, dv \tag{2.24}$$

$$\Psi(d) = \frac{\delta}{\varepsilon} e^{-U(d)/\delta^2} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} v e^{-v^2/2} Q(d, v)\, dv \tag{2.25}$$

3. MAIN RESULTS

The main result of the paper is the leading-order approximation to the diffusion process (2.1) with absorbing boundaries at spatial locations $x = 0$ and $x = d$, and trajectories originating ar $x = 0$. We find that in the high-friction limit the Langevin process (2.1) becomes the Smoluchowski process

$$\frac{x}{d\hat{t}} = -U'(x) + \sqrt{2}\delta \frac{dw}{d\hat{t}} \tag{3.1}$$

with trajectories originating at $x = 0$, and absorbed at points

$$x_R^* = d - \frac{\delta}{\beta} \zeta\left(\frac{1}{2}\right) - \frac{1}{2\beta^2} U'(d) + O\left(\frac{1}{\beta^3}\right) \tag{3.2}$$

$$x_L^* = \frac{\delta}{\beta} \left[\zeta\left(\frac{1}{2}\right) + \mathscr{B}_0 \right] + O\left(\frac{1}{\beta^2}\right) \tag{3.3}$$

Here $\zeta(1/2) = -1.460435...$ is the Riemann zeta function, and $\mathscr{B}_0$ is defined by the series in Eq. (5.36). In particular, if the velocity source is Maxwelllian, then $\mathscr{B}_0$ is given by Eq. (5.41) and its numerical value is

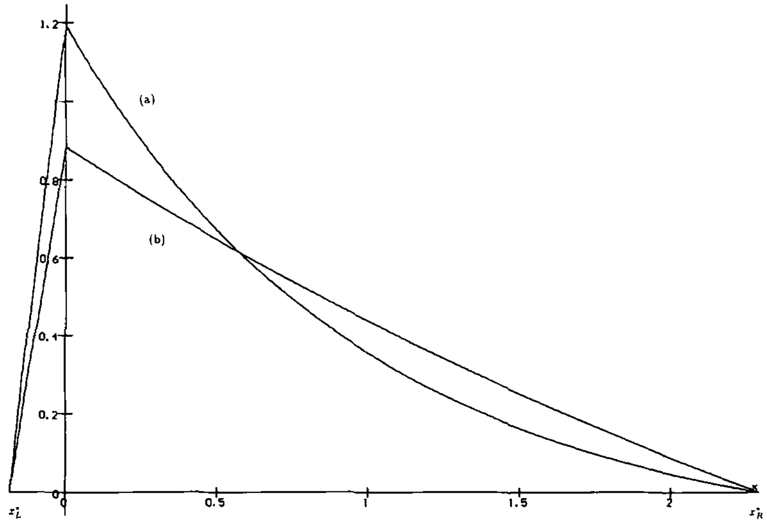

Fig. 2. The outer marginal steady-state density $P^{\mathrm{OUT}}_{\mathrm{mar}}(x)$ in the case of a linear potential $U(x)=x$ and (a) $\beta=5$, $\delta=1$, (b) $\beta=10$, $\delta=2$, with $d=2$. $P^{\mathrm{OUT}}_{\mathrm{mar}}(d)=O(1/\beta)$.

$\mathscr{B}_0=0.52424\ldots$. Thus the transition probability density function (see Fig. 2) of the process $x(t)$ satisfies the Fokker–Planck equation

$$\frac{\partial p}{\partial \hat{t}}=\frac{\partial}{\partial x}\left[U'(x)\,p\right]+\delta^2\frac{\partial^2 p}{\partial x^2} \tag{3.4}$$

with the initial condition

$$\lim_{\hat{t}\to 0} p(x, \hat{t} \mid \xi)=\delta(x) \tag{3.5}$$

and the absorbing boundary conditions

$$p(x^*_R, \hat{t} \mid \xi)=0, \qquad p(x^*_L, \hat{t} \mid \xi)=0 \tag{3.6}$$

We find that the solution to the problem (2.10) uniformly valid on $(0, d]$ is given by

$$P(x, v)=\frac{\beta}{(2\pi)^{1/2}}\,e^{-v^2/2\delta^2}\,e^{-U(x)/\delta^2}\,Q(x, v) \tag{3.7}$$

Here

$$Q(x, v)=Q^{\mathrm{OUT}}(x, v)+Q^{\mathrm{L}}_{\mathrm{BL}}(x, v)+Q^{\mathrm{R}}_{\mathrm{BL}}(x, v) \tag{3.8}$$

where

$$Q^{\mathrm{OUT}}(x, v) = C\left[\int_0^x e^{U(\tilde{x})/\delta^2}\, d\tilde{x} - \frac{\delta}{\beta} v e^{U(x)/\delta^2}\right] + D + O\left(\frac{1}{\beta^2}\right) \tag{3.9}$$

$$Q^{\mathrm{L}}_{\mathrm{BL}}(x, v) = \sum_{n=1}^{\infty} A_n^- e^{\lambda_n^-(a^0)\beta x/\delta}\, V_n^-\left(\frac{v}{\delta}, a^0\right) \tag{3.10}$$

$$Q^{\mathrm{R}}_{\mathrm{BL}}(x, v) = M \sum_{n=1}^{\infty} \frac{1}{n\mathcal{N}'(\lambda_n^-(a^d))}\, e^{\lambda_n^-(a^d)\,\beta(d-x)/\delta}\, V_n^-\left(\frac{v}{\delta}, a^d\right) \tag{3.11}$$

In the case of the Maxwellian velocity source the constants in Eqs. (3.9)–(3.11) are given by

$$C = \frac{1}{\beta\delta} e^{U(0)/\delta^2} \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^d e^{U(\tilde{x})/\delta^2}\, d\tilde{x}} + O\left(\frac{1}{\beta^2}\right) \tag{3.12}$$

$$D = -\frac{1}{\beta\delta} e^{U(0)/\delta^2}\left[\zeta\left(\frac{1}{2}\right) + \mathscr{B}_0\right] + O\left(\frac{1}{\beta^2}\right) \tag{3.13}$$

$$M = -\frac{1}{\beta^2} e^{U(d)/\delta^2} e^{U(0)/\delta^2} \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^d e^{U(\tilde{x})/\delta^2}\, d\tilde{x}} + O\left(\frac{1}{\beta^3}\right) \tag{3.14}$$

and the definitions of the eigenvalues λ_n^-, eigenfunctions V_n^-, and the function $\mathcal{N}$ are given in Eqs. (4.6) and (4.39). The coefficients A_n^- are given by Eq. (5.34), and are $O(1/\beta)$. Using (3.7) in Eqs. (2.13) and (2.17), we find the mfpt of the trajectories out of the strip D is given by

$$E\tau = (E\tau)^{\mathrm{OUT}} + (E\tau)^{\mathrm{L}}_{\mathrm{BL}} + (E\tau)^{\mathrm{R}}_{\mathrm{BL}} \tag{3.15}$$

where $(E\tau)^{\mathrm{OUT}}$ denotes the contribution to the mfpt due to the outer solution $Q^{\mathrm{OUT}}(x, v)$ and is given by

$$(E\tau)^{\mathrm{OUT}} = -\frac{\zeta(1/2) + \mathscr{B}_0}{\delta \int_0^d e^{U(\tilde{x})/\delta^2}\, d\tilde{x}} e^{U(0)/\delta^2} \int_0^d e^{-U(x)/\delta^2}\left[\int_x^d e^{U(\tilde{x})/\delta^2}\, d\tilde{x}\right] dx + O\left(\frac{1}{\beta}\right) \tag{3.16}$$

while $(E\tau)^{\mathrm{L}}_{\mathrm{BL}}$ and $(E\tau)^{\mathrm{R}}_{\mathrm{BL}}$ denote the contributions to the mfpt due to the boundary layer terms $Q^{\mathrm{L}}_{\mathrm{BL}}$ and $Q^{\mathrm{R}}_{\mathrm{BL}}$, respectively. This contributions are $O(1/\beta)$ and $O(1/\beta^2)$, respectively. They are given by Eqs. (5.58) and (5.59). We find that the probability flux at d is given by

$$\Psi(d) = -\frac{\delta}{\beta} e^{U(0)/\delta^2} \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^d e^{U(\tilde{x})/\delta^2}\, d\tilde{x}} + O\left(\frac{1}{\beta^2}\right) \tag{3.17}$$

Equations (3.15) and (3.17), which are obtained from the solution to the full two-dimensional problem, give the same expressions for the mfpt and the probability flux at the boundaries as the Smoluchowski approximation (3.1) considered on the interval (x_L^*, x_R^*).

4. IMPORTANT TECHNICAL ANALYSES AND RESULTS

In this section we derive results which will be used later to write the uniform solution to Eq. (2.22). In Section 5 we will show that the boundary layer analyses of Eq. (2.22) (about $x=0$ and $x=d$) lead to boundary value problems which can be written as

$$Q_{vv} + (-v+a)\, Q_v - vQ_{\tilde{x}} = 0 \tag{4.1}$$

$$Q(\tilde{x}, v>0) = 0$$

$$Q \approx \text{const}_1 + \text{const}_2\, e^{-a(v-\tilde{x})} \qquad \text{as} \quad \tilde{x} \to \infty$$

where the constant a depends on the particular boundary larger [see Eqs. (5.12) and (5.18)].

4.1. Operator $\mathscr{L}$ and Its Properties

Separation of variables in Eq. (4.1) leads to an eigenvalue problem for the operator $\mathscr{L}$,

$$\mathscr{L}V \equiv V'' + (-v+a)\, V' = \lambda v V \tag{4.2}$$

Two solutions to Eq. (4.2) which have the desired behavior for $|v| \gg 1$ are given by[27]

$$V_1(v) = e^{(v-a)^2/4}\, U(\tfrac{1}{2} + \lambda^2 - \lambda a,\ v - a + 2\lambda) \tag{4.3}$$

$$V_2(v) = e^{(v-a)^2/4}\, U(\tfrac{1}{2} + \lambda^2 - \lambda a,\ -v + a - 2\lambda) \tag{4.4}$$

where U denotes the parabolic cylinder function of index $\frac{1}{2} + \lambda^2 - \lambda a$. The two solutions are linearly independent if

$$\lambda^2 - a\lambda = n \qquad \text{for} \quad n = 0, 1, 2, \ldots \tag{4.5}$$

Therefore we obtain the set of eigenvalues and eigenfunctions of the problem (4.2),

$$\lambda_0(a) \equiv \lambda_0 = 0 \leftrightarrow V_0(v, a) = 1 \tag{4.6}$$

$$\tilde{\lambda}_0(a) \equiv \tilde{\lambda}_0 \equiv a \leftrightarrow \tilde{V}_0(v, a) = e^{-av}$$

$$\lambda_n^+(a) \equiv \lambda_n^+ \equiv [a + (a^2 + 4n)^{1/2}]/2 \leftrightarrow V_n^+(v, a) = \mathrm{He}_n(v - a + 2\lambda_n^+)\, e^{-\lambda_n^+(v-a) - (\lambda_n^+)^2}$$

$$\lambda_n^-(a) \equiv \lambda_n^- \equiv [a - (a^2 + 4n)^{1/2}]/2 \leftrightarrow V_n^-(v, a) = \mathrm{He}_n(v - a + 2\lambda_n^-)\, e^{-\lambda_n^-(v-a) - (\lambda_n^-)^2}$$

Here He_n denotes the Hermite polynomials.

We observe that the operator $\mathscr{L}$ is self-adjoint in the inner product

$$\langle V, W \rangle \equiv \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-(v-a)^2/2}\, V(v)\, W(v)\, dv \tag{4.7}$$

and that the eigenfunctions ϕ_m, ϕ_n of $\mathscr{L}$ corresponding to distinct eigenvalues λ_m, λ_m ($\mathscr{L}\phi_m = \lambda_m v \phi_m$) are orthogonal in the inner product

$$\langle v\phi_m, \phi_n \rangle = 0 \qquad \text{if} \quad n \neq m \tag{4.8}$$

The set $\{V_0, \tilde{V}_0, V_n^\pm, n = 1, 2, \ldots\}$ is an orthogonal basis of the square-integrable functions in the inner product (4.7). Thus any such a function f can be expanded as

$$f(v) = I_0 V_0 + \tilde{I}_0 \tilde{V}_0 + \sum_{n=1}^{\infty} (I_n^+ V_n^+ + I_n^- V_n^-) \tag{4.9}$$

where

$$I_0 = \frac{\langle vf, V_0 \rangle}{\langle v V_0, V_0 \rangle}; \qquad \tilde{I}_0 = \frac{\langle vf, \tilde{V}_0 \rangle}{\langle v \tilde{V}_0, \tilde{V}_0 \rangle}, \qquad I_n^\pm = \frac{\langle vf, V_n^\pm \rangle}{\langle v V_n^\pm, V_n^\pm \rangle} \tag{4.10}$$

Using properties of Hermite polynomials, we find the normalization constants .

$$\langle v V_0, V_0 \rangle = a$$

$$\langle v \tilde{V}_0, \tilde{V}_0 \rangle = -a$$

$$\langle v V_n^\pm, V_n^\pm \rangle = (a - 2\lambda_n^\pm)\, n! \tag{4.11}$$

In particular, if the source density is Maxwellian, then the analysis near the location of the source will require the expansion of $f(v) = 1/v$ in terms of the eigenfunctions. We have

$$\frac{1}{v} = \frac{1}{a} - \frac{1}{a} e^{-a^2/2} e^{-av} + \sum_{n=1}^{\infty} \frac{(\lambda_n^+)^n e^{-(\lambda_n^+)^2/2}}{(a - 2\lambda_n^+)\, n!} V_n^+(v) + \sum_{n=1}^{\infty} \frac{(\lambda_n^-)^n e^{-(\lambda_n^-)^2/2}}{(a - 2\lambda_n^-)\, n!} V_n^-(v) \tag{4.12}$$

If f and h are two square-integrable functions with respect to the inner product (4.7), then we have

$$\langle vf, h \rangle = \frac{\langle vf, V_0 \rangle \langle vh, V_0 \rangle}{\langle v V_0, V_0 \rangle} + \frac{\langle vf, \tilde{V}_0 \rangle \langle vh, \tilde{V}_0 \rangle}{\langle v \tilde{V}_0, \tilde{V}_0 \rangle} + \sum_{n=1}^{\infty} \frac{\langle vf, V_n^+ \rangle \langle vh, V_n^+ \rangle}{\langle v V_n^+, V_n^+ \rangle} + \sum_{n=1}^{\infty} \frac{\langle vf, V_n^- \rangle \langle vh, V_n^- \rangle}{\langle v V_n^-, V_n^- \rangle} \tag{4.13}$$

We will need the following identities, which follow directly from the properties of Hermite polynomials.[27] We define the nonweighted inner product

$$\langle V, W \rangle_{NW} \equiv \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} e^{-v^2/2}\, V(v)\, W(v)\, dv = e^{a^2/2} \langle V e^{-av}, W \rangle \tag{4.14}$$

Then we have for $n = 1, 2, \ldots$

$$\langle V_n^\pm, 1 \rangle_{\mathrm{NW}} = (\lambda_n^\pm - a)^n\, e^{-[(\lambda_n^\pm)^2 - 2\lambda_n^\pm a]/2} \tag{4.15}$$

$$\langle V_n^\pm, v \rangle_{\mathrm{NW}} = e^{a^2/2} \langle V_n^\pm, v e^{-av} \rangle = 0 \tag{4.16}$$

$$\langle V_n^\pm, v^2 \rangle_{\mathrm{NW}} = -a(\lambda_n^\pm - a)^{n-1}\, e^{-[(\lambda_n^\pm)^2 - 2\lambda_n^\pm a]/2} \tag{4.17}$$

Equations (4.16) and (4.17) imply that

$$a \langle V_n^\pm, 1 \rangle_{\mathrm{NW}} = \lambda_n^\mp \langle V_n^\pm, v^2 \rangle_{\mathrm{NW}} \tag{4.18}$$

4.2. Solution to the Problem (4.1)

In this section we adapt the method of ref. 20 to solve the half-range boundary value problem (4.1). In ref. 20 a problem analogous to (4.1) with $a \equiv 0$ was solved. Below we present the main steps in the generalization of the method of ref. 20 to the case of $a \neq 0$.

The eigenfunction expansion of (4.1) is given by

$$Q(\tilde{x}, v) = C_0 + \tilde{C}_0\, e^{-a(v-\tilde{x})} + \sum_{n=1}^{\infty} C_n^+ e^{\lambda_n^+ \tilde{x}} V_n^+(v) + \sum_{n=1}^{\infty} C_n^- e^{\lambda_n^- \tilde{x}} V_n^-(v) \tag{4.19}$$

In order to satisfy the matching conditions, we must have $C_n^+ = 0$ for $n = 1, 2,\ldots$.

Next we take the Laplace transform of (4.19) with respect to $\tilde{x}$ to obtain

$$\hat{Q}(s, v) = \frac{C_0}{s} + \frac{\tilde{C}_0}{s-a} e^{-av} + \sum_{n=1}^{\infty} \frac{C_n^-}{s-\lambda_n^-} V_n^-(v) \tag{4.20}$$

We also take the Laplace transform of the problem (4.1) with respect to $\tilde{x}$, solve it, and compare the solution with (4.20). We have

$$\hat{Q}_{vv} + (-v+a)\,\hat{Q}_v - sv\hat{Q} = 0 \qquad \text{for} \quad v > 0 \tag{4.21}$$

whose solution is

$$\hat{Q}(s, v) = e^{(v-a)^2/4}\, U(v-a+2s)\, F(s) \tag{4.22}$$

where U denotes the parabolic cylinder function with index $\frac{1}{2} + s^2 - sa$, and the function $F(s)$ is determined below. Since for $v > 0$, Eqs. (4.20) and (4.22) represent the same function, the two representations must have the same singularities in s. So we have

$$F(s) = \frac{E(s)}{s(s-a)\,\mathcal{N}(s)} \tag{4.23}$$

where

$$\mathcal{N}(s) = \prod_{n=1}^{\infty} c_n(s - \lambda_n^-) \tag{4.24}$$

with coefficients c_n chosen so the product (4.24) converges, and it has a suitable behavior for $s \gg 1$, and $\mathcal{N}(s=0) = 1$ (see Section 4.4); and $E(s)$ is an entire function of s. In order to determine $E(s)$ we analyze (4.22) for $s \gg 1$. We have

$$U(v-a+2s) \sim (2\pi)^{1/2} \hat{s}^{1/3} \exp[-\tfrac{1}{2}\hat{s}^2 + (\hat{s}^2 - \tfrac{1}{4}a^2) \ln \hat{s}]\, \mathrm{Ai}(\hat{s}^{1/3} v)[1 + O(\hat{s}^{-2/3})] \tag{4.25}$$

where Ai denotes the Airy function, and

$$\hat{s} \equiv s - \frac{a}{2} \tag{4.26}$$

Combining Eq. (4.25) with Eq. (4.38) in Eq. (4.22), we obtain that for $s \gg 1$

$$\hat{Q}(s, v) \sim \frac{E(s)}{s^{1+1/6}} \tag{4.27}$$

Since $\hat{Q}$ must be meromorphic in s, the only entire function which can satisfy (4.27) is $E(s) \equiv E \equiv \text{const}$. Next we invert the Laplace transform (4.22) to obtain

$$Q(\tilde{x}, v) = E\left[-\frac{1}{a} + \frac{1}{a\mathcal{N}(a)} e^{-a(v-\tilde{x})} + \sum_{n=1}^{\infty} \frac{e^{\lambda_n^- \tilde{x}}}{n\mathcal{N}'(\lambda_n^-)} V_n^-(v, a) \right] \tag{4.28}$$

This is the solution to problem (4.1).

4.3. Expansion of V_m^+

In this section we solve the problem

$$\begin{aligned} &Q_{vv}^{(m)} + (-v+a)\, Q_v^{(m)} - vQ_{\tilde{x}}^{(m)} = 0 \\ &\qquad Q^{(m)}(\tilde{x}=0, v) = V_m^+(v) \qquad \text{if} \quad v > 0 \\ &\qquad Q \approx \text{const} \qquad \text{as} \quad \tilde{x} \to \infty \end{aligned} \tag{4.29}$$

We write the solution to (4.29) in the form of the expansion

$$Q^{(m)}(\tilde{x}, v) = C_0^{(m)} + \sum_{n=1}^{\infty} C_n^{-(m)} e^{\lambda_n^- \tilde{x}} V_n^-(v) \tag{4.30}$$

We compare the Laplace transform of (4.30) (with respect to the variable $\tilde{x}$) with the solution to the Laplace transform of the original problem (4.29), that is, with the solution to the problem

$$\hat{Q}_{vv}^{(m)}(s, v) + (-v+a)\, \hat{Q}_v^{(m)}(s, v) - vs\hat{Q}^{(m)}(s, v) + vV_m^+(s, v) = 0 \qquad \text{if} \quad v > 0 \tag{4.31}$$

As before, the two forms of solutions must have the same singularities in s, so we find that

$$\hat{Q}^{(m)} = \frac{V_m^+(v)}{s - \lambda_m^+} + \frac{e^{(v-a)^2/4}\, U(v - a + 2s)\, E(s)}{s(s - \lambda_m^+)\, \mathcal{N}(s)} \tag{4.32}$$

By the same argument as before we find that $E(s) \equiv E \equiv \text{const}$. We determine the value of E so Eq. (4.32) has no singularity at $s = \lambda_m^+$. We have

$$E = -\lambda_m^+ \mathcal{N}(\lambda_m^+) \tag{4.33}$$

We invert the Laplace transform (4.32) to find

$$Q^{(m)}(x, v) = \beta_0^{(m)} + \sum_{n=1}^{\infty} \beta_n^{(m)} e^{\lambda_n^- \tilde{x}} V_n^-(v) \tag{4.34}$$

where

$$\begin{aligned} \beta_0^{(m)} &= \mathcal{N}(\lambda_m^+) \\ \beta_n^{(m)} &= \frac{\mathcal{N}(\lambda_m^+)}{\lambda_n^-(\lambda_m^+ - \lambda_n^-)\,\mathcal{N}'(\lambda_n^-)} \end{aligned} \tag{4.35}$$

Since the function (4.34) is the solution to the problem (4.29), in particular for $v > 0$ and $\tilde{x} = 0$ we obtain the expansion of V_m^+ in $\{1, V_n^-, n = 1, 2,...\}$

$$V_m^+(v, a) = \beta_0^{(m)} + \sum_{n=1}^{\infty} \beta_n^{(m)} V_n^-(v, a) \qquad \text{if} \quad v > 0 \tag{4.36}$$

4.4. The Function $\mathcal{N}(s)$

We need to define a function $\mathcal{N}(s)$ in the form of the product (4.24), so it has simple zeros at $\lambda_n^-(a) \equiv [a - (4n + a^2)^{1/2}]/2$. We observe that the function

$$\begin{aligned} N(s) = \prod_{k=1}^{\infty} \exp\left\{-2\hat{s}\left[\left(k + \frac{a^2}{4}\right)^{1/2} - \left(k - 1 + \frac{a^2}{4}\right)^{1/2}\right]\right\} \\ \times \left(\frac{k + 1 + a^2/4}{k + a^2/4}\right)^{\hat{s}^2/2} \left(1 + \frac{\hat{s}}{(k + a^2/2)^{1/4}}\right) \end{aligned} \tag{4.37}$$

satisfies this condition. Here $\hat{s}$ is given by (4.26). Moreover, the function $N(s)$ for $s \gg 1$ has the asymptotic behavior given by

$$N(s) \sim \hat{s}^{-1/2} \exp\left[-\frac{\hat{s}^2}{2} + \left(\hat{s}^2 - \frac{a^2}{2}\right) \ln \hat{s}\right] \tag{4.38}$$

which is the necessary condition to establish (4.27). Thus we define

$$\mathcal{N}(s) = \frac{N(s)}{N(0)} \tag{4.39}$$

so $\mathcal{N}(0) = 1$.

To calculate the asymptotic behavior of the formulas derived in Sections 5 and 6 we need the following asymptotic expansion of $\mathcal{N}$ for $s \ll 1$:

$$\mathcal{N}(s) = 1 + s\zeta\left(\frac{1}{2}\right) + \frac{s^2}{2}\zeta^2\left(\frac{1}{2}\right) + O(s^3) \tag{4.40}$$

Here $\zeta(\frac{1}{2})$ denotes the Riemann zeta function.

5. PROBLEM ON $(0, d)$

5.1. Free-Space Operators

We seek an approximate solution to the Langevin equation for long time $\hat{t} = O(1)$, $t = O(\beta)$. We expand the solution to Eq. (2.8) in the form of the asymptotic series[2,3]

$$p(x, v, \hat{t}) = p^0(x, v, \hat{t}) + \frac{1}{\beta}p^1(x, v, \hat{t}) + \frac{1}{\beta^2}p^2(x, v, \hat{t}) + \dots \tag{5.1}$$

The leading-order solution is given by

$$p^0 = \frac{1}{(2\pi)^{1/2}\delta}e^{-v^2/2\delta^2}P^0(x, \hat{t}) \tag{5.2}$$

where P^0 satisfies

$$\frac{\partial P^0}{\partial \hat{t}} = L_x P^0 \equiv \delta^2 \frac{\partial^2 P^0}{\partial x^2} + \frac{\partial}{\partial x}(U'(x)\, P^0) \tag{5.3}$$

On the time scale $\hat{t}$ the position x and the velocity v of the diffusion process (x, v) become independent processes, with the velocity following the Maxwellian (thermal equilibrium) distribution, while the position x satisfies the stochastic Smoluchowski differential equation

$$\frac{dx}{d\hat{t}} = -U'(x) + \sqrt{2}\delta\frac{dw}{d\hat{t}} \tag{5.4}$$

5.2. Outer Solution

Away from the boundaries and the source, at a distance greater than $O(\varepsilon)$, Eq. (2.22) reduces to the outer problem

$$\mathcal{L}_{x,v}Q^{\mathrm{OUT}} = 0 \tag{5.5}$$

We seek the solution to Eq. (5.5) in the form of the asymptotic series

$$Q^{\mathrm{OUT}}(x, v) = Q^0(x, v) + \varepsilon Q^1(x, v) + \varepsilon^2 Q^2(x, v) + \dots \tag{5.6}$$

Upon substituting Eq. (5.6) into Eq. (5.5) we obtain the infinite system of equations

$$Q^0_{vv} - vQ^0_v = 0$$
$$Q^i_{vv} - vQ^i_v = \delta v Q^{i-1}_x - \frac{U'(x)}{\delta} Q^{i-1}, \qquad i = 1, 2, \ldots \tag{5.7}$$

The system is solved recursively to yield

$$Q^{\mathrm{OUT}}(x, v) = C\left[\int_0^x e^{U(\tilde{x})/\delta^2}\, d\tilde{x} - \varepsilon\delta v e^{U(x)/\delta^2} + \varepsilon^2 \frac{v^2}{2} U'(x)\, e^{U(x)/\delta^2}\right] + D + O(\varepsilon^3) \tag{5.8}$$

where C and D are arbitrary constants. If the potential $U(x)$ is linear, that is, if $U(x) = U_0 x$, then we sum up the series (5.8) to obtain the exact solution to Eq. (5.5) given by

$$Q^{\mathrm{OUT}}(x, v) = C_1 e^{U_0(x/\delta^2 - \varepsilon v/\delta)} + D_1 \tag{5.9}$$

5.3. Local Solution about $x = d$

We introduce a local variable $y \equiv (d - x)/\varepsilon$ in Eq. (2.22) to obtain the problem for the inner function Q^R given by

$$Q^{\mathrm{R}}_{vv} + \left(-v + \varepsilon \frac{U'(d)}{\delta}\right) Q^{\mathrm{R}}_v + \delta v Q^{\mathrm{R}}_y = 0 \tag{5.10}$$
$$Q^{\mathrm{R}}(y = 0, v < 0) = 0$$
$$Q^{\mathrm{R}}(y, v) \to Q^{\mathrm{OUT}}(y, v) \qquad \text{as} \quad y \to \infty$$

According to the analysis of Section 4, the eigensolution to Eq. (5.10) is given by

$$Q^{\mathrm{R}}(y, v) = M\left[d_0 + \tilde{d}_0 e^{a^d(v + y/\delta)} + \sum_{n=1}^{\infty} d_n^- e^{\lambda_n^- y/\delta} V_n^-(-v, a^d)\right] \tag{5.11}$$

where

$$\lambda_n^- \equiv \lambda_n^-(a^d), \qquad a^d = -\varepsilon \frac{U'(d)}{\delta}$$
$$d_0 = -\frac{1}{a^d}, \qquad \tilde{d}_0 = \frac{1}{a^d \mathcal{N}(a^d)}, \qquad d_n^- = \frac{1}{n\mathcal{N}'(\lambda_n^-)} \tag{5.12}$$

and the function $\mathcal{N}(s)$ is defined by Eq. (4.39); the constant M is determined below.

5.4. Local Solution About $x = 0$

We introduce a local variable $z \equiv x/\varepsilon$ in Eq. (2.22) to obtain the problem for the inner function Q^L given by

$$Q^{\mathrm{L}}_{vv} + \left(-v + \varepsilon \frac{U'(0)}{\delta}\right) Q^{\mathrm{L}}_v - \delta v Q^{\mathrm{L}}_z \approx 0 \tag{5.13}$$

with the boundary condition

$$Q^{\mathrm{L}}(z = 0^-, v > 0) = 0 \tag{5.14}$$

and the jump condition

$$Q^{\mathrm{L}}(z = 0^-, v > 0) - Q^{\mathrm{L}}(z = 0^+, v > 0) = \frac{\varepsilon}{\delta} \frac{s(v)}{v} e^{U(0)/\delta^2} \tag{5.15}$$

and the matching condition

$$Q^{\mathrm{L}}(z, v) \to Q^{\mathrm{OUT}}(z, v) \qquad \text{as} \quad z \to \infty \tag{5.16}$$

According to the analysis of Section 4, the solution to Eq. (5.13) which can satisfy Eq. (5.16) is given by

$$Q^{\mathrm{L}^+}(z, v) = A_0 + \tilde{A}_0 e^{-a^0(v - z/\delta)} + \sum_{n=1}^{\infty} A_n^- e^{\lambda_n^- z/\delta} V_n^-(v, a^0) \qquad \text{for} \quad z > 0 \tag{5.17}$$

where

$$\lambda_n^- \equiv \lambda_n^-(a^0), \qquad a^0 = \varepsilon \frac{U'(0)}{\delta} \tag{5.18}$$

and where A_0, $\tilde{A}_0$, and A_n^-, $n = 1, 2, \ldots$, are arbitrary constants. In order to satisfy the jump condition (5.15) we consider the problem (5.13) for $z < 0$ on an interval of an infinitesimal length. On such an interval the solution does not decay or does not grow significantly. Moreover, there are no matching conditions imposed on the left for $z < 0$. Therefore the eigenfunction expansion to Eq. (5.13) must include the entire set of eigenfunctions. We have

$$Q^{\mathrm{L}^-}(z, v) = B_0 + \tilde{B}_0 e^{-a^0(v - z/\delta)} + \sum_{n=1}^{\infty} B_n^- e^{\lambda_n^- z/\delta} V_n^-(v, a^0) + \sum_{n=1}^{\infty} B_n^+ e^{\lambda_n^+ z/\delta} V_n^+(v, a^0) \tag{5.19}$$

The jump condition (5.15) gives the relation between A's and B's. We expand the right-hand side of Eq. (5.15) in the eigenfunctions $\{V_0, \tilde{V}_0, V_n^{\pm}, n=1,2,\ldots\}$ as

$$\frac{\varepsilon}{\delta}\frac{s(v)}{v}e^{U(0)/\delta^2}=I_0V_0(v)+\tilde{I}_0\tilde{V}_0(v)+\sum_{n=1}^{\infty}I_n^-V_n^-(v)+\sum_{n=1}^{\infty}I_n^+V_n^+(v) \quad (5.20)$$

to obtain

$$\begin{aligned} A_0-B_0&=I_0\\ \tilde{A}_0-\tilde{B}_0&=\tilde{I}_0\\ A_n^- -B_n^-&=I_n^-\\ -B_n^+&=I_n^+ \end{aligned} \quad (5.21)$$

However, as shown in Section 4, the function which satisfies Eq. (5.13) together with the boundary conditions (5.14) must have the form

$$Q^{\mathrm{L}}(z,v)=K\left[\alpha_0+\tilde{\alpha}_0e^{-a^0(v-z/\delta)}+\sum_{n=1}^{\infty}\alpha_n^-e^{\lambda_n^- z/\delta}V_n^-(v,a^0)\right] \quad (5.22)$$

where K is a constant to be determined, and

$$\begin{gathered} \lambda_n^-\equiv\lambda_n^-(a^0),\qquad a^0=\varepsilon\frac{U'(0)}{\delta}\\ \alpha_0=-\frac{1}{a^0},\qquad \tilde{\alpha}_0=\frac{1}{a^0\mathcal{N}(a^0)},\qquad \alpha_n^-=\frac{1}{n\mathcal{N}'(\lambda_n^-)} \end{gathered} \quad (5.23)$$

The function $\mathcal{N}(s)$ is defined in Eq. (4.39). Next we rewrite Eq. (5.19) in the expansion (4.34). We obtain

$$\begin{aligned} Q^{\mathrm{L}^-}(z,v)=&\,B_0+\tilde{B}_0e^{a^0(v-z/\delta)}+\sum_{m=1}^{\infty}B_m^+e^{\lambda_m^+z/\delta}\beta_0^{(m)}\\ &+\sum_{n=1}^{\infty}V_n^-(v)\left\{B_n^-e^{\lambda_n^-z/\delta}+\sum_{m=1}^{\infty}B_m^+e^{\lambda_m^+z/\delta}\beta_n^{(m)}\right\}\qquad \text{if}\quad v>0 \end{aligned} \quad (5.24)$$

The two expansions (5.24) and (5.22) must agree at $z=0$. This comparison leads to the relations between the coefficients B's and α's. We have

$$\begin{aligned} B_0&=K\alpha_0-\sum_{m=1}^{\infty}B_m^+\beta_0^{(m)}\\ \tilde{B}_0&=K\tilde{\alpha}_0\\ B_n^-&=K\alpha_n^--\sum_{m=1}^{\infty}B_m^+\beta_n^{(m)} \end{aligned} \quad (5.25)$$

5.5. Uniform Solution

All unknown constants at this point of the analysis are to be determined from matching conditions. The matching condition in (5.10) gives

$$C\int_0^d e^{U(\tilde{x})/\delta^2}\,d\tilde{x} + D = M[d_0 + \tilde{d}_0]$$

$$Ce^{U(d)/\delta^2} = MU'(d)\frac{\tilde{d}_0}{\delta^2} \tag{5.26}$$

while matching between Q^{L^+} and Q^{OUT} according to (5.16) gives

$$A_0 + \tilde{A}_0 = D$$

$$Ce^{U(0)/\delta^2} = \tilde{A}_0\frac{U'(0)}{\delta^2} \tag{5.27}$$

Equations (5.21) and (5.25)–(5.27) determine all constants uniquely. We have

$$C = \frac{\alpha_0\tilde{I}_0 - \tilde{\alpha}_0(I_0 + \sum_{m=1}^{\infty} I_m^+\beta_0^{(m)})}{\alpha_0\delta^2 e^{U(0)/\delta^2}/U'(0) - \rho\tilde{\alpha}_0} \tag{5.28}$$

$$K = \frac{\rho\tilde{I}_0 - \delta^2 e^{U(0)/\delta^2}(I_0 + \sum_{m=1}^{\infty} I_m^+\beta_0^{(m)})/U'(0)}{\alpha_0\delta^2 e^{U(0)/\delta^2}/U'(0) - \rho\tilde{\alpha}_0} \tag{5.29}$$

$$A_0 = C\rho \tag{5.30}$$

$$D = C(\rho + \delta^2 e^{U(0)/\delta^2}/U'(0)) \tag{5.31}$$

$$M = C\delta^2 e^{U(d)/\delta^2}/U'(d)\,\tilde{d}_0 \tag{5.32}$$

$$\tilde{A}_0 = C\delta^2 e^{U(0)/\delta^2}/U'(0) \tag{5.33}$$

$$A_n^- = I_n^- + K\alpha_n^- + \sum_{m=1}^{\infty} I_m^+\beta_n^{(m)} \tag{5.34}$$

and with K given by Eq. (5.29), all B's are directly calculated from Eq. (5.25). The coefficient ρ in the above equations is defined as

$$\rho = \delta^2[e^{U(d)/\delta^2}(1 + d_0/\tilde{d}_0)/U'(d) - e^{U(0)/\delta^2}/U'(0)] - \int_0^d e^{U(\tilde{x})/\delta^2}\,d\tilde{x} \tag{5.35}$$

We observe that all constants, and thus the solution to the problem, depend on the density of the velocity source through the cofficients $I_0, \tilde{I}_0, I_n^{\pm}$. We define

$$\mathscr{B}_0 = \sum_{m=1}^{\infty} I_m^+ \beta_0^{(m)} \tag{5.36}$$

In particular, if the source density is Maxwellian, (2.6), then in the limit of $\varepsilon \ll 1$ we have

$$C = \varepsilon \frac{e^{U(0)/\delta^2}}{\delta} \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^d e^{U(\tilde{x})/\delta^2}\, d\tilde{x}} + O(\varepsilon^2) \tag{5.37}$$

$$D = -\varepsilon \frac{e^{U(0)/\delta^2}}{\delta} \left[\zeta\left(\frac{1}{2}\right) + \mathscr{B}_0 \right] + O(\varepsilon^2) \tag{5.38}$$

$$K = \frac{\varepsilon}{\delta} e^{U(0)/\delta^2} + O(\varepsilon^2) \tag{5.39}$$

$$M = -\varepsilon^2 e^{U(d)/\delta^2} e^{U(0)/\delta^2} \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^d e^{U(\tilde{x})/\delta^2}\, d\tilde{x}} + O(\varepsilon^3) \tag{5.40}$$

and so on. Here ζ denotes the Riemann zeta function $[\zeta(1/2) = -1.46035...]$ and

$$\mathscr{B}_0 \equiv \sum_{n=1}^{\infty} \frac{n^{n/2} e^{-n/2} \mathscr{N}(\sqrt{n})}{2\sqrt{n}\, n!} \tag{5.41}$$

The function $\mathscr{N}$ in Eq. (5.41) is calculated by substituting $a \equiv 0$ into Eq. (4.39). We evaluate the series (5.41) numerically to find

$$\mathscr{B}_0 = 0.52424... \tag{5.42}$$

Thus the uniform solution to the problem (2.22) is given by

$$Q(x, v) = Q^{\mathrm{OUT}}(x, v) + M \sum_{n=1}^{\infty} d_n^- e^{\lambda_n^-(a^d)(d-x)/\varepsilon\delta} V_n^-(-v, a^d) + \sum_{n=1}^{\infty} A_n^- e^{\lambda_n^-(a^0)x/\varepsilon\delta} V_n^-(v, a^0) \tag{5.43}$$

In particular, in the case of a linear potential we write the uniform solution as

$$Q(x, v) = C_1 e^{U_0(x/\delta^2 - \varepsilon v/\delta)} + D_1 + K^d \sum_{n=11}^{\infty} s_n^- e^{\lambda_n^-(-a)(d-x)/\delta\varepsilon} V_n^-(-v, -a)$$

$$+ \sum_{n=1}^{\infty} A_n^- e^{\lambda_n^-(a)x/\delta\varepsilon} V_n^-(v, a) \tag{5.44}$$

Here $a \equiv \varepsilon U_0/\delta$ and

$$K^d = \frac{\varepsilon}{d} \mathcal{N}(-a) \frac{\mathcal{N}(a) e^{-a^2/2} - 1 - a \sum_{n=1}^{\infty} I_n^+ \beta_0^{(m)}}{\mathcal{N}(a) e^{-U_0 d/\delta^2} - \mathcal{N}(-a)} \tag{5.45}$$

To leading order the constant K^d is given by

$$K^d = \varepsilon^2 \frac{U_0 \zeta(1/2) + \mathscr{B}_0}{\delta^2 e^{-U_0 d/\delta^2} - 1} + O(\varepsilon^3) \tag{5.46}$$

The other constants are given by

$$C_1 = -\frac{1}{a\mathcal{N}(-a)} e^{U_0 d/\delta^2} K^d = -\frac{\varepsilon}{\delta} e^{-U_0 d/\delta^2} \frac{\zeta(1/2) + \mathscr{B}_0}{e^{-U_0 d/\delta^2} - 1} + O(\varepsilon^2) \tag{5.47}$$

$$D_1 = \frac{1}{a} K^d = \frac{\varepsilon}{\delta} \frac{\zeta(1/2) + \mathscr{B}_0}{e^{-U_0 d/\delta^2} - 1} + O(\varepsilon^2) \tag{5.48}$$

The coefficients A_n^- are given by Eq. (5.34) with K replaced by K^d. They are $O(\varepsilon)$. We observe that Eq. (5.44) gives the solution to the problem (2.22) with a transcendentally small error.

The marginal density of the position x at $x = d$ does not vanish, but rather takes on a value $O(\varepsilon)$ (cf. Fig. 2). We have

$$P_{\text{mar}}(d) = \frac{1}{\varepsilon} e^{-U(d)/\delta^2} \langle Q(d, v), 1 \rangle_{\text{NW}} = \varepsilon \frac{[\zeta(1/2) + \mathscr{B}_0]\, \zeta(1/2)}{\int_0^d e^{U(x)/\delta^2}\, dx} e^{U(0)/\delta^2} + O(\varepsilon^2) \tag{5.49}$$

We calculate the flux at $x = d$ according to the formula (2.25). We have

$$\Psi(d) = \frac{\delta}{\varepsilon} e^{-U(d)/\delta^2} \langle Q(d, v), v \rangle_{\text{NW}} = -\varepsilon\delta \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^d e^{U(x)/\delta^2}\, dx} e^{U(0)/\delta^2} + O(\varepsilon^2) \tag{5.50}$$

Equivalently, the probability that a trajectory exits the strip at $x=0$ is given by

$$\Psi(0) = 1 + \varepsilon\delta \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^d e^{U(x)/\delta^2}\, dx} e^{U(0)/\delta^2} + O(\varepsilon^2) \tag{5.51}$$

As the width of the strip d goes to infinity, the trajectory returns to the origin with probability one if and only if $\int_0^\infty e^{U(x)/\delta^2}\, dx = \infty$. Otherwise the trajectory returns to the origin with probability given by

$$\Psi(0) = 1 + \varepsilon\delta \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^\infty e^{U(x)/\delta^2}\, dx} e^{U(0)/\delta^2} + O(\varepsilon^2) \tag{5.52}$$

In the case of a linear potential the result (5.52) can be deduced from the analysis in ref. 19.

For the clarity of the presentation we calculate the mfpt in the case of the linear potential. Analogous results hold in the case of an arbitrary potential. By employing the formula (5.44) in Eq. (2.23), we find

$$E\tau = (E\tau)^{\text{OUT}} + (E\tau)^{\text{L}}_{\text{BL}} + (E\tau)^{\text{R}}_{\text{BL}} \tag{5.53}$$

Here $(E\tau)^{\text{OUT}}$ denotes the contribution to the mfpt due to the outer solution $Q^{\text{OUT}}(x, v)$. It is given by

$$\begin{aligned}(E\tau)^{\text{OUT}} &= \frac{1}{\varepsilon}\left[C_1 e^{\varepsilon^2 U_0^2/2\delta^2} d + D_1 \frac{\delta^2}{U_0}\left(1 - e^{-U_0 d/\delta^2}\right)\right] \\ &\approx -\frac{\delta}{U_0} \frac{[\zeta(1/2) + \mathscr{B}_0](-d + \int_0^d e^{U_0 x/\delta^2}\, dx)}{\int_0^d e^{U_0 x/\delta^2}\, dx} + O(\varepsilon)\end{aligned} \tag{5.54}$$

The contributions to the mfpt due to the boundary layer terms on the left and on the right are denoted by $(E\tau)^{\text{L}}_{\text{BL}}$ and $(E\tau)^{\text{R}}_{\text{BL}}$, respectively. We have

$$(E\tau)^{\text{R}}_{\text{BL}} = \frac{1}{\varepsilon} K^d \int_0^d e^{-U_0 x/\delta^2} \left(\sum_{n=1}^\infty d_n^- \langle V_n^-(v), 1\rangle_{\text{NW}}\, e^{\lambda_n^-(-a)(d-x)/\varepsilon\delta}\right) dx \tag{5.55}$$

which is given, with a transcendentally small error, by

$$\begin{aligned}(E\tau)^{\text{R}}_{\text{BL}} &= \delta e^{U_0 d/\delta^2} K^d \sum_{n=1}^\infty d_n^- \frac{\langle V_n^-, 1\rangle_{\text{NW}}}{\lambda_n^+(a)} \\ &= \frac{\delta}{a} e^{U_0 d/\delta^2} K^d \sum_{n=1}^\infty d_n^- \langle V_n^-, v^2\rangle_{\text{NW}} \\ &= -\delta K^d(\mu + O(\varepsilon)) \\ &= \varepsilon^2 \frac{U_0 \zeta(1/2) - \mathscr{B}_0}{e^{U_0 d/\delta^2} - 1} \mu + O(\varepsilon^3)\end{aligned} \tag{5.56}$$

where

$$\mu \equiv \sum_{n=1}^{\infty} \frac{n^{n/2} e^{-n/2} \mathcal{N}(\sqrt{n})}{2n \cdot n!} \approx 0.206198... \tag{5.57}$$

[Here $\mathcal{N}$ is evaluated according to Eq. (4.39) with $a \equiv 0$.] In the case of an arbitrary potential, $(E\tau)_{\rm BL}^{\rm R}$ is given by

$$(E\tau)_{\rm BL}^{\rm R} = \varepsilon^2 \delta e^{U(0)/\delta^2} \frac{\zeta(1/2) + \mathscr{B}_0}{\int_0^d e^{U(\tilde{x})/\delta^2}\, d\tilde{x}} \mu \tag{5.58}$$

For a linear potential, similar calculations give

$$(E\tau)_{\rm BL}^{\rm L} = \delta \sum_{n=1}^{\infty} A_n^- \frac{\langle V_n^-, 1 \rangle_{\rm NW}}{\lambda_n^+(a)} = O(\varepsilon) \tag{5.59}$$

In the case of an arbitrary potential the boundary layer solution at the left gives the same contributions as given in Eq. (5.59), multiplied by $e^{U(0)/\delta^2}$, and with a replaced by $\varepsilon U'(0)/\delta$. The contribution due to the outer solution in general is given by

$$(E\tau)^{\rm OUT} = \frac{1}{\varepsilon} \int_0^d e^{-U(x)/\delta^2} \langle Q^{\rm OUT}(x, v), 1 \rangle_{\rm NW}\, dx \tag{5.60}$$

which is $O(1)$.

5.6. Extrapolation Length

In this section we derive the reduced boundary value problem corresponding to (2.10) in the asymptotic limit $\beta \gg 1$. As shown in Section 5.1, the Fokker–Planck operator $L_{x,v}$ for the diffusion process (2.1) approaches the forward operator L_x for the Smoluchowski problem in the limit as $\beta \to \infty$. Thus Eq. (2.10) becomes

$$L_x P \equiv \delta^2 \frac{\partial^2 P}{\partial x^2} + \frac{\partial}{\partial x} (U'(x) P) = -\delta(x - 0^+) \tag{5.61}$$

Next we derive the boundary conditions for Eq. (5.61) consistent with the absorbing boundary conditions (2.11) and (2.12). In particular, Eq. (5.61) equipped with these new boundary conditions must give the same values of the mean passage time and the probability flux at the boundary, among others, as those obtained from the full two-dimensional problem in Eqs. (2.13), (2.16), (2.17). The outer solution to Eq. (2.10) according to (2.9) and (2.21) is given by

$$P^{\mathrm{OUT}}(x, v) = \frac{1}{(2\pi)^{1/2}} e^{-v^2/2} e^{-U(x)/\delta^2} Q^{\mathrm{OUT}}(x, v) \tag{5.62}$$

where Q^{OUT} is given by Eq. (5.8). Hence the marginal density $P^{\mathrm{OUT}}_{\mathrm{mar}}(x)$ is given by

$$\begin{aligned} P^{\mathrm{OUT}}_{\mathrm{mar}}(x) &\equiv \int_{-\infty}^{\infty} P^{\mathrm{OUT}}(x, v)\, dv \\ &= e^{-U(x)/\delta^2} \left[C \left(\int_0^x e^{U(\tilde{x})/\delta^2}\, d\tilde{x} + \frac{\varepsilon^2}{2} U'(x)\, e^{U(x)/\delta^2} \right) + D \right] + O(\varepsilon^3) \end{aligned} \tag{5.63}$$

To leading order the constants C and D are given by Eqs. (5.37) and (5.38), respectively. $P^{\mathrm{OUT}}_{\mathrm{mar}}(x)$ vanishes at a point x^*_{R} which is the solution to

$$C \left[\int_0^{x^*_{\mathrm{R}}} e^{U(\tilde{x})/\delta^2}\, d\tilde{x} + \frac{\varepsilon^2}{2} U'(x^*_{\mathrm{R}})\, e^{U(x^*_{\mathrm{R}})/\delta^2} \right] + D \approx 0 \tag{5.64}$$

It is given by

$$x^*_{\mathrm{R}} \approx d + \frac{\delta^2}{U'(d)} \ln \left[\mathcal{N} \left(-\varepsilon \frac{U'(d)}{\delta} \right) - \frac{\varepsilon^2}{2\delta^2} U'^2(d) \right] \tag{5.65}$$

Expanding $\mathcal{N}$ for $\varepsilon \ll 1$, as in Eq. (4.40), we find that

$$x^*_{\mathrm{R}} = d - \varepsilon\delta\zeta(\tfrac{1}{2}) - \tfrac{1}{2}\varepsilon^2 U'(d) + O(\varepsilon^3) \tag{5.66}$$

We observe that in the case of a linear potential the formula (5.65) is exact, that is, the sign $\approx$ should be replaced with the equality sign. Since the (outer) marginal density vanishes at x^*_{R}, It is the point of extrapolated absorbing conditions for the operator (5.61).

In order to determine the point of the extrapolated boundaries on the left (near the location of the source), we proceed as follows. We extend the problem (5.61) to the left of the location of the source, that is, for $x < 0$. We require that for $x < 0$ the density P satisfies the problem

$$L_x P = 0 \tag{5.67}$$

and it is continuous at x, that is,

$$P(0) = P_{\text{mar}}^{\text{OUT}}(0) \tag{5.68}$$

and Eq. (5.61) determines the jump in the derivative as

$$\frac{d}{dx} P_{\text{mar}}^{\text{OUT}}(0) - \frac{d}{dx} P(0) = -\frac{1}{\delta^2} \tag{5.69}$$

We solve the problem (5.67)–(5.69) to obtain

$$P(x) = e^{-U(x)/\delta^2} \left[\left(\frac{1}{\delta^2} + C \right) \int_0^x e^{U(t)/\delta^2} \, dt + D \right] \qquad \text{for} \quad x < 0 \tag{5.70}$$

Next we calculate the point x_{L}^* where P vanishes. Its coordinate is given by

$$x_{\text{L}}^* \approx \frac{\delta^2}{U'(0)} \ln \left[1 - \frac{D}{C + 1/\delta} \frac{U'(0)}{\delta^2} e^{-U(0)/\delta^2} \right] \tag{5.71}$$

Using Eqs. (5.37) and (5.38), we find that

$$x_{\text{L}}^* = \varepsilon\delta(\zeta(\tfrac{1}{2}) + \mathscr{B}_0) + O(\varepsilon^2) \tag{5.72}$$

where $\mathscr{B}_0$ is given by Eq. (5.41). Therefore the problem for the Smoluchowski equation (5.61) equipped with zero boundary conditions at points x_{R}^* and x_{L}^*, given by (5.66) and (5.72), respectively, is the leading-order approximation to the problem (2.10) in the asymptotic limit of large friction.

6. EXAMPLE: NO-FORCE CASE

In this section we show how our analysis reduces to the case of no external force, that is, if $U(x) \equiv 0$ on $[0, d]$. For the clarity of the presentation we assume that the velocity source at $x = 0$ is Maxwellian, (2.6). Thus we solve the problem

$$\begin{gathered} \mathscr{L}_{x,v} \equiv Q_{vv} - vQ_v - \varepsilon\delta v Q_x = -\varepsilon^2 \delta(x - 0^+) \\ Q(x = 0^-, v > 0 \mid \xi = 0^+) = 0 \\ Q(x = d, v < 0 \mid \xi = 0^+) = 0 \end{gathered} \tag{6.1}$$

In the analysis below we use the properties of the operator $\mathscr{L}_{x,v}$ derived in ref. 20. The uniform solution to the problem (6.1) is

$$Q(x, v) = C(x - \varepsilon\delta v) + D + \sum_{n=1}^{\infty} A_n^- V_n^-(v)\, e^{-n^{1/2}x/\varepsilon\delta}$$

$$- K^d \sum_{n=1}^{\infty} \beta_n V_n^+(v)\, e^{-n^{1/2}(d-x)/\varepsilon\delta} \qquad \text{if} \quad 0 < x \leqslant d \qquad (6.2)$$

where

$$C = -\frac{\varepsilon}{\delta(d + 2\varepsilon\delta\alpha)} [\alpha - \mathscr{B}_0]$$

$$D = \frac{\varepsilon(d + \varepsilon\delta\alpha)}{\delta(d + 2\varepsilon\delta\alpha)} [\alpha - \mathscr{B}_0]$$

$$K^d = \frac{\varepsilon^2}{d + 2\varepsilon\delta\alpha} [\alpha - \mathscr{B}_0]$$

$$K^0 = \frac{\varepsilon}{\delta(d + 2\varepsilon\delta\alpha)} [d + \varepsilon\delta\alpha + \varepsilon\delta\mathscr{B}_0]$$

$$A_n^- = \frac{\varepsilon}{\delta}\left[(-1)^{n+1}\beta_n\left(1 - \frac{\varepsilon\delta(\alpha - \mathscr{B}_0)}{d + 2\varepsilon\delta\alpha}\right) + \sum_{k=1}^{\infty} I_k^+ \beta_n^{(k)} + I_n^-\right] \qquad (6.3)$$

with

$$\alpha = \left|\zeta\left(\frac{1}{2}\right)\right|, \qquad \beta_n = \frac{N(\sqrt{n})}{2n!\,\sqrt{n}}, \qquad \beta_k^{(n)} = \frac{(-1)^k \sqrt{n}\, N(\sqrt{n})\, N(\sqrt{k})}{2k!\,(\sqrt{k} + \sqrt{n})} \qquad (6.4)$$

and

$$N(s) = \prod_{n=1}^{\infty} \left(1 + \frac{s}{\sqrt{n}}\right)\left(\frac{k+1}{k}\right)^{s^2/2} e^{-2s[n^{1/2} - (n-1)^{1/2}]} \qquad (6.5)$$

$$I_n^+ = -\frac{\varepsilon}{\delta}\frac{n^{n/2} e^{-n/2}}{2n!\,\sqrt{n}}, \qquad I_n^- = -(-1)^n I_n^+ \qquad (6.6)$$

Next we calculate properties of the process (x, v). The probability that a trajectory leaves D at $x = d$, given it started at $x = 0^+$ according to formula (2.25), is given by

$$\Psi(d) = \frac{\delta}{\varepsilon}\int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{1/2}}\, v e^{-v^2/2} Q(d, v)\, dv = \frac{\delta}{\varepsilon} K^d = \frac{\varepsilon\delta}{d + 2\varepsilon\delta\alpha}(\alpha - \mathscr{B}_0) \qquad (6.7)$$

Similarly, the probability that a trajectory that originates at $x=0^+$ leaves D at $x=0$ (with $v<0$) is given by

$$\Psi(0) = -\frac{\delta}{\varepsilon}\int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{1/2}} ve^{-v^2/2} Q(0^-, v)\,dv = \frac{\delta}{\varepsilon} K^0 = \frac{d+\varepsilon\delta(\alpha+\mathscr{B}_0)}{d+2\varepsilon\delta\alpha} \tag{6.8}$$

Moreover, the probability $\Psi^+(0)$ that a trajectory which originates at $x=0$ with $v>0$ leaves D at $x=0$ with $v<0$ is given by

$$\Psi^+(0) \equiv \Psi(0) - \frac{1}{2} \equiv \frac{1}{2} - \Psi(d) = \frac{d+2\varepsilon\delta\mathscr{B}_0}{2(d+2\varepsilon\delta\alpha)} \tag{6.9}$$

The mfpt to the boundaries of D, according to Eq. (2.33), is given by, with a transcendentally small error,

$$E\tau = \frac{1}{\varepsilon}\left[C\frac{d^2}{2} + Dd + \varepsilon\delta \sum_{n=1}^{\infty} A_n^- \langle V_n^-, 1\rangle - \varepsilon\delta K^d \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \beta_n \langle V_n^+, 1\rangle \right] \tag{6.10}$$

Here $\langle V_n^+, 1\rangle \equiv n^{n/2} e^{-n/2}$ and $\langle V_n^-, 1\rangle \equiv (-1)^n \langle V_n^+, 1\rangle$. To leading order Eq. (6.10) becomes

$$E\tau = \frac{d(\alpha-\mathscr{B}_0)}{2\delta} + O(\varepsilon) \tag{6.11}$$

The leading-order outer approximation to the problem (2.10) with $U(x)\equiv 0$ is given by

$$\frac{1}{\delta^2} P_{xx} = -\frac{1}{\varepsilon}\delta(x)$$
$$P(x_L^*) = 0, \qquad P(x_R^*) = 0 \tag{6.12}$$

where $x^*_{L(R)}$ denote the coodinates of the extrapolated boundary conditions, which are found by the mthod of Section 4.4 to be

$$x_L^* = -\varepsilon\delta(\alpha-\mathscr{B}_0), \qquad x_R^* = d+\varepsilon\delta\alpha \tag{6.13}$$

The outer problem (6.12) gives formulas for the mfpt and the probability of exit points the same as those given by Eqs. (6.11) and (6.7)–(6.9).

7. DISCUSSION

In this paper we have presented a systemtic asymptotic analysis of a boundary value problem for the Fokker–Planck operator corresponding to the Langevin equation in the high-friction limit. We assumed that the trajectories of the process originate with an arbitrary velocity density. Absorbing boundary conditions were prescribed at the spatial location $x=0$ of the velocity source and at another location, $x=d \gg O(1)$ away. The global effects of a potential function on the probability flux at the boundaries and the mean time to absorption were included. The friction parameter β was assumed to be large compared to other parameters of the problem: the temperature (noise strength) δ^2, and the force at the boundaries, $U'(0)$ and $U'(d)$. More specifically, the outer expansion (in the case of an arbitrary potential) and the boundary layer analysis are valid as long as $\beta \gg \delta$. For clarity of presentation we showed results in Section 3 in the high-damping regime with the other parameters being $O(1)$. However, our analysis yields more general results. It shows the interplay between parameters of the problem: β, δ, and the strength of the force at the boundaries, $U'(d)$ and $U'(0)$. The extrapolation length depends on the factor $a^d \equiv U'(d)/\beta\delta$ through the function $\mathcal{N}(-a^d)$—compare Eq. (5.65). Results shown in Section 3 were obtained by expanding the function $\mathcal{N}$ for small values of its argument. That is, if force is $O(1)$ at the boundary, then this expansion is valid for $\beta \gg 1/\delta$. However, the examples of Section 3 are derived from the more general results obtained in Section 5 under the assumption that $\beta \gg \delta$.

The half-range expansion calculations presented here are based on a modification of the method of refs. 20 and 21 to include effects of the force on the boundary layer solution. An alternative method for treatment of half-range expansion boundary value problems in the case of a constant force and a half-space domain $x>0$ was developed in refs. 18 and 19. Our boundary layer solution at $x=d$ corresponds to the analysis of the *Milne problem* of refs. 18 and 19, while our boundary layer solution at $x=0$ correspond to the *albedo problem* of refs. 18 and 19. In particular, our asymptotic limit of high damping [with other parameters being $O(1)$] in the boundary layer solution corresponds to the small-force field limit of ref. 14. [In ref. 14 the distinguished limit of the damping coefficient and the noise strength $\beta\delta^2 = O(1)$ was considered.] In this asymptotic limit both methods yield identical results for the behavior of the boundary layer solutions.

To summarize the contributions of this paper: we have extended the analysis of refs. 14, 18, and 19 to include the effects of both a finite slab geometry and a general potential. [The restriction $d \gg O(1)$ may be removed by applying the expansion method of Section 5.1 to both boundaries simultaneously.] These generalizations enable the calculation of the mean time to absorption, which is $O(1)$. Thus the effect of the boundary layer next to the source is singular. [If the trajectories originate away from the boundaries, then in the Smoluchowski limit the mean time to absorption is $O(\beta)$.] We have calculated the probabilities of absorption at each of the boundaries and the velocity distributions of these exit points. If the width of the slab becomes infinite, we have calculated the probability of recurrence for an arbitrary potential, whic agrees with the results of ref. 19 in the case of a linear potential.

A more detailed discussion of the extrapolation length as a function of the parameters of the problem will be given elsewhere.[25] We will generalize the analysis to include the effects of the noncharacteritic boundaries. We will also generalize the method of refs. 20 and 21 to handle half-range expansion problems for other types of differential scattering operators.

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